

# Wave Height and Horizon Dip

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**ABSTRACT:** *A mariner who takes the height of the sun or a star to find his position at sea, must correct his observation for horizon dip. Throughout history, dip values have been tabulated based on the idealized assumption of a perfectly flat sea. Literature on wave height correction for dip is scarce, especially in the Western part of the world and suggested procedures are often conflicting or incorrect. We investigate the effect of wave height on the apparent horizon for realistic wave height distributions by optical ray tracing methods and show that there are two eye height corrections: one representing the apparent rise of the horizon itself and the second when sights are selectively taken from wave crests. Tables and formulas for dip and horizon distance taking wave height into account are presented. Copyright © 2015 Institute of Navigation.*

## INTRODUCTION

Finding one's position - or rather position line - by measuring the height of a celestial body, one must use in some form, via tabulation or directly, the cosine rule which states:

$$\begin{aligned} \cos(ZEN) = & \sin(LAT)\sin(DEC) \\ & + \cos(LAT)\cos(DEC)\cos(LHA) \end{aligned} \quad (1)$$

where  $LAT$  is the observer's latitude,  $DEC$  the declination of the sun or star,  $LHA$  the local hour angle, and  $ZEN$  the zenith angle of the heavenly body.

When using a sextant, or in olden days, a cross-staff, the mariner does not measure the zenith angle but instead the height above the apparent horizon. In the above formula he then wants to replace  $\cos(ZEN)$  by  $\sin(ALT)$ , where  $ALT$  is the celestial body's altitude,  $90^\circ - ZEN$ , but before doing so he must correct his measured height for dip: the angular distance by which the apparent horizon, "the rim of heaven," is below the true horizontal.

With reference to Figure 1, let the observer's elevation above the flat sea be  $h_0$ . Near-horizontal light rays have a radius of curvature of normally about  $6 \times$  the Earth's radius,  $R$ .

It is easily shown that in first approximation:

$$\begin{aligned} \text{dip} = & \sqrt{(1 - R/r) \frac{2h_0}{R}} \text{ and without refraction, } r \rightarrow \infty : \\ \text{dip} = & \sqrt{\frac{2h_0}{R}} \end{aligned} \quad (2)$$

Likewise it follows that the distance to the horizon is given by:

$$\begin{aligned} \text{dist. horizon} = & \sqrt{\frac{2h_0 R}{1 - R/r}} \text{ and for straight rays :} \\ \text{dist. horizon} = & \sqrt{2h_0 R} \end{aligned} \quad (3)$$

Navigators are used to reckoning dip in arcminutes ( $'$ ), and distances in nautical miles (nm). The Earth's radius is  $R = 6356766$  m, and ignoring the ray curvature one has:

$$\begin{aligned} \text{dip} = & 1'.93 \sqrt{h_0} \text{ and} \\ \text{dist. horizon} = & 1.93 \sqrt{h_0} \text{ nm} \end{aligned} \quad (4)$$

where the elevation above the water surface,  $h_0$ , stands in meters.

The fact that the multiplying factor, 1.93, is the same in both cases comes as no surprise since one minute of arc along a great circle equals a nautical mile by definition.

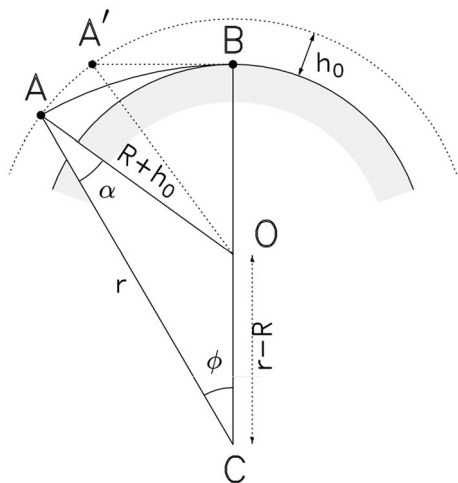


Fig. 1– Schematic illustration of horizon dip and distance. In the absence of refraction, the observer is in A' and a light ray drawn from his eye to the right skims the sea surface in B. Refraction causes the ray to be curved downward and for the ray to still go through the horizon at B, the observer will have to distance himself backward to A. At the same time the downward tilt of the ray, which is the horizon dip, has become less.

Since 2008 the *Nautical Almanac* [1] has based its refraction and horizon dip tables on the so-called Modified US1976 standard atmosphere [2], wherein temperature and pressure at sea level have been adopted as 10 °C and 1013.25 hPa, respectively. Horizontal and near-horizontal light rays then possess a radius of curvature  $r = 5.71 R$  and upon inserting this into Eqs. (2) and (3), the expressions in Eq. (4) modify into:

$$\begin{aligned} \text{dip} &= 1'.75\sqrt{h_0} \text{ and} \\ \text{dist. horizon} &= 2.12 \sqrt{h_0} \text{ nm} \end{aligned} \quad (5)$$

Thus the effect of the light's curvature is to lower dip and at the same time to push the horizon farther out. The tables of the *Nautical Almanac* are consistent with the expressions in Eq. (5) and agree also with older tabulations showing at most a minimal difference in the second decimal of the multiplying factors 1.75 and 2.12. The formulas of Eq. (5) apply to a perfectly flat sea.

But waves shift the visible separation between sea and sky upward. In the words of Henry Raper, in the 1857- and later editions of his book *The practice of Navigation and Nautical Astronomy* [3]: "Since the sea-horizon is formed by the eminences of the waves, it should be higher in bad weather."

This is naively illustrated in Figure 2. The dashed curve represents a light ray traced backward from the sailor's eye. It just skims the flat sea to the right and thus defines what he sees as the horizon. With waves present, the visual separation between water and sky is raised: the solid curve that just passes over the distant wave crests lies higher than the

dashed line. At the same time the apparent horizon has come nearer.

Imagine that - not quite realistic but useful for the sake of argument - all waves are equally high. Wave height,  $H$ , is defined as the height difference between a wave crest and an adjacent trough. Crests are raised over the median or flat-sea surface by approximately  $H/2$ . This is exactly the situation for sinusoidal waves and still close for wind driven waves where crests are narrower than troughs. Ignoring the ship's own up and down movement, or averaging over it, this simplified model suggests that Eq. (5) be replaced by:

$$\begin{aligned} \text{dip} &= 1'.75\sqrt{h_0 - H/2} \text{ and} \\ \text{dist. horizon} &= 2.12 \sqrt{h_0 - H/2} \text{ nm} \end{aligned} \quad (6)$$

Such is the advice, given by Krasavtsev and Klyustin in their book "Nautical Astronomy" [4], which is the English translation of the 1970 edition of "Morehodnaya Astronomiya." The same prescription is given by Kazanskii [5] in his book "Terrestrial Refraction over Extended Water Surfaces" (1966). In the later editions of the *Morehodnaya Astronomiya*, 1978 and 1986, by Krasavtsev alone [6, 7], this has been changed to subtracting 1/3 of the wave height. The recommendation is meant for big ships and eye height is taken relative to the median sea level. In all these cases, the proposed eye height corrections appear to be empirical ones and they are presented without derivation or reference.

In the 'official' nautical literature from the West, we have not been able to locate any reference to the effect of wave height, other than Raper's above mentioned remark. The modern *Bowditch* [8] mentions the wave height correction in its glossary, but fails to discuss it in the actual text.

For yachtsmen, eye height is already low and it is common practice to take sights when the ship rides the top of a wave, so one gets a better view of the horizon. In books by yachtsmen, written for yachtsmen, one encounters different opinions. Tom Cunliffe [9] advises to *add* half a wave height, because that is by how much the wave has lifted

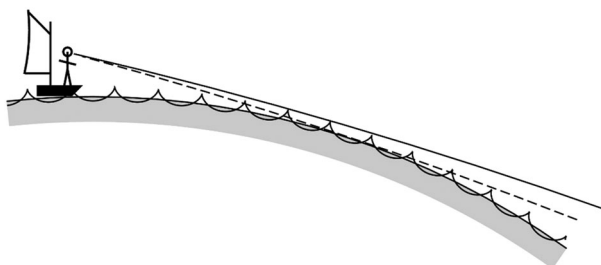


Fig. 2– Waves raise the apparent horizon and bring it nearer.

you upward. John Karl [10] rightfully rejects this procedure, because when on top of a wave, your horizon will in turn be made up by other distant waves. And he suggests to make no correction at all.

Without making any calculation yet, we can argue that the correct procedure is to indeed *reduce* one's eye height by a fraction of the wave height, when looking up dip from a tabulation, and that this fraction should be even larger than half the significant wave height.

In Figure 3, three scenarios are sketched: in the middle a narrow channel of a perfectly flat sea stretches out to the horizon and the observer is situated 5 m above it. To the left, waves are all of the same height, rising 2 m above median sea level (a significant wave height of 4 m). The apparent horizon lies higher than it is seen above a flat sea and the observer will wish to subtract half a wave height from his usual eye height before entering his dip tables, as suggested in Eq. (6).

To the right, wave heights follow a realistic distribution that will be specified in the following section, and have been randomly generated from this distribution. Although 86% of them are lower than the waves to the left, still 14% will be higher. Near the horizon, waves cannot be distinguished individually and it is the higher waves that determine what is seen as the horizon, just as Henry Raper formulated more than a century and a half ago. The important thing to note here is that the waves from the realistic distribution produce a higher horizon than the waves to the left, which are all of the same height.

Just by how much the horizon is raised is the topic of the following sections, where we will also discuss the scenario that the navigator will try to selectively take his sight when riding a wave top.

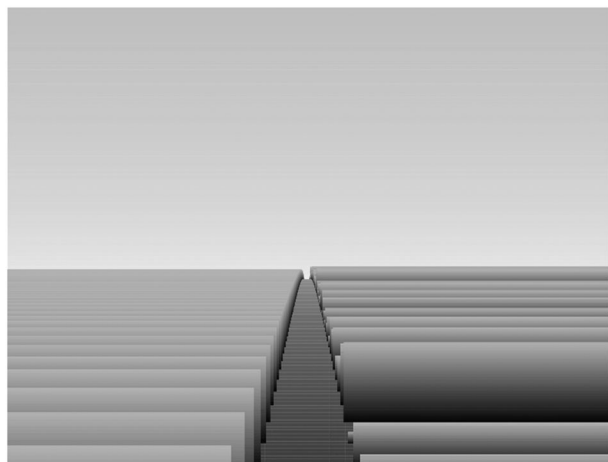


Fig. 3– Illustration of the effect of wave height on dip. Left: all waves are of equal height above the median sea level. Middle: flat sea. Right: random wave heights from a realistic wave height distribution and with the same significant wave height as the sea to the left.

## REALISTIC WAVE HEIGHT DISTRIBUTIONS

Over the past half century wave analysis has become a research topic of growing importance and applicability. For the following we refer, without an attempt at completeness, to the books of Tucker and Pitt [11], Holthuijsen [12], Young [13], Groen and Dorrestein [14] and to the "Guide to Wave Analysis and Forecasting" of the WMO (World Meteorological Organization) [15].

Non-composite wind-driven waves are found to follow a Rayleigh distribution:

$$P(H) = \frac{4H}{H_s^2} \exp \left[ -2 \left( \frac{H}{H_s} \right)^2 \right] \quad (7)$$

where  $H_s$  is the so-called *significant wave height*.

The probability of encountering a wave higher than  $H$ , which is the definite integral from  $H$  to infinity over  $P(H)$ , is:

$$P(\text{wave height} > H) = \exp \left[ -2(H/H_s)^2 \right] \quad (8)$$

Based on this formula, significant wave height is often identified with the average of the largest 30% of the wave heights.

In the present work we are especially interested in the height distribution of the wave crests above the median sea level, which to a good approximation is  $h = H/2$ . Its distribution is:

$$P(h) = 2P(2h) = \frac{16h}{H_s^2} \exp \left[ -8 \left( \frac{h}{H_s} \right)^2 \right] \quad (9)$$

In addition we will need an appropriate 'random number generator' for this distribution function. By analogy to the work of Box and Muller [16], who present such an algorithm for a random normal variate, Eq. (9) is found to be reproduced by:

$$\frac{H_s}{4} \sqrt{-2 \ln[\text{rnd}(1)]} \quad (10)$$

where  $\text{rnd}(1)$  is a random number, taken from a 'white' distribution between 0 and 1.

Patterns of wind-driven waves depend on water depth, wind speed, the undisturbed length of the wind path, called the *fetch*, and the time during which the waves build up under an unchanged wind, the *duration*. In the present work we restrict ourselves to a *fully developed sea* over deep water. That is: we assume a fetch and duration long enough

for the wave height and length distributions to build up to a steady pattern.

Groen and Dorrestein [14] show in their Diagram 1 that under these conditions the significant wave height approaches  $H_s \rightarrow 0.24U^2/g$ , where  $U$  is the wind speed in m/sec, as measured at 10 m above sea level and  $g=9.81\text{ m/sec}^2$  is the gravitational acceleration. From the same diagram one finds that the wave period, the time lapse between two successive crests, tends to  $T \rightarrow 2\pi U/g$ . The same diagram may also be found on p. 44 of ref. [15]. This limit for  $T$  relies on the assumption that in the end, the phase velocity of the waves, which is given by  $C = \sqrt{gL/2\pi}$ , will tend to equal the wind speed  $U$ , and of course on the relation between wave period and wave length,  $L = CT$ .

Putting all this together, one finds that the ratio between the significant wave height and the wave length, the steepness, approaches  $H_s/L \rightarrow 0.24/2\pi = 0.038 = 1/26.2$ , independent of wind speed.

Stated in a different way: on deep water and under a constant wind blowing over a long enough stretch of water for sufficient duration, the wave length tends to 26x the significant wave height.

Plausible and elegant as this result may be, the assumption that wind speed and wave velocity will become equal remains an assumption and in the available literature one can encounter different limiting values for the steepness. Tucker and Pitt [10] give 1/19.7 and from numbers given by Holthuijsen [12] one gets 1/39.2. Pierson and Moskowitz [17] obtain limits, also quoted in [12], that give a steepness of 1/33.8. They use, however, wind speeds measured at 19.5 m instead of the more conventional 10 m above the water.

## FINDING THE DISTRIBUTION OF DIP

The path of a near-horizontal light ray is given, to very good approximation, by the equation [18]:

$$d^2h/dx^2 = c = 1/R - 1/r \quad (11)$$

Here,  $h$  is the height above the median sea surface and  $x$  is the distance traveled along it. The observer is at  $x=0$  and his eye height will be  $h_0$ .  $R$  is the Earth's radius and  $r$  is the ray's radius of curvature, which may be considered constant as long as the ray does not deviate too much from horizontal. For the standard atmosphere, adopted by the *Nautical Almanac* (10 °C, 1013.25 hPa) we have  $r=5.71R$ .

With  $c$  constant, the solution of Eq. (11) is:

$$h = h_0 + \beta x + \frac{1}{2}cx^2 \quad (12)$$

where  $\beta$  is the tilt angle of the ray at the observer's position. In the equation,  $\beta$  stands in radians, but it may be rewritten to arcminutes by multiplying it with a factor  $60 \times 180/\pi$ .

For a given value of  $H_s$ , the significant wave height, we adopt a wave length  $L=26.2 H_s$ , in accordance with the steepness as found from Groen and Dorrestein [14]. The effect of choosing a different steepness will be evaluated and discussed at a later stage. With the algorithm of Eq. (10) we then adopt a random wave height in each point  $x=nL$ , all independent of one another, for  $n=1,2,\dots$ . Next, we find the tilt  $\beta$  for each of these wave tops. The highest, that is, the least negative value thus found for  $\beta$ , defines the ray that just misses all the waves and which therefore represents the visual separation

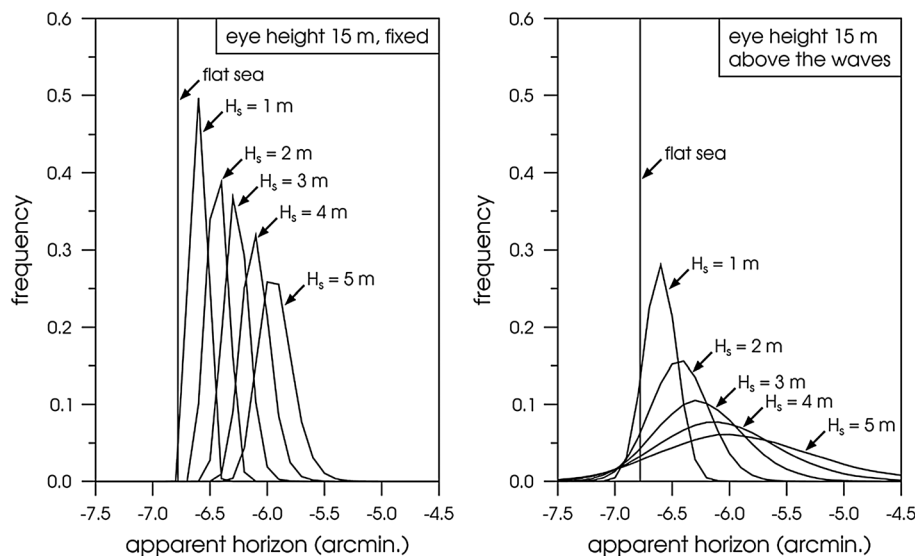


Fig. 4– Left: distributions of horizon dip for different significant wave heights as seen from a fixed eye height of 15 m above the median sea level. Right: distributions for an eye height of 15 m above the actual water level, moving up and down with the waves.

between water and sky. This procedure is then repeated many times, typically one thousand to ten thousand times, until a clear statistical distribution of these  $\beta$ -values is achieved. Checks against full ray tracing calculations with the methods described in references [2] and [18] are in full agreement, which justifies the use of the simpler but very much faster method described above.

## DIP DISTRIBUTIONS

Horizon dip distributions obtained in the above described way are presented in Figure 4 for an eye height of 15 m. We distinguish two scenarios. First, the eye height is kept fixed above the median sea level, such as it would be for a flat sea. The distributions are close to Gaussian and shift to the right with increasing wave height. Waves are seen to lift the apparent horizon and thus decrease dip. The distributions exhibit a well-pronounced maximum, which represents the *most probable dip value*.

In the right hand panel of Figure 4 the corresponding distributions are shown for the scenario that the ship itself moves up and down with the full amplitude of the waves. The procedure of ray tracing is still the same but eye height is now chosen as:

$$h'_0 = h_0 + \frac{H_s}{2} \sqrt{-2 \ln(a)} \cdot \cos[2\pi b] \quad (13)$$

where  $a$  and  $b$  are two independent  $\text{rnd}(1)$  random numbers. The additional term is the Box-Muller algorithm [16] for a random variate of normal distribution and an rms-width equal to  $H_s$ . The effect of adding this Gaussian component to the eye height is a broadening of the distributions, which does not come as a surprise. The broadening does not alter the peak positions of the distributions. This is understandable, because they are just convolutions of the original distribution with a Gaussian of mean zero and the mean of a convolution equals the sum of the means of the individual constituting distributions.

It should be remarked here that a ship with a bridge height of 15 m, as chosen in the example of

Figure 4, is a pretty big ship and the up- and down movement of eye height will be of lesser amplitude than that of the water surface itself. The broadening will thus be less than in the extreme case, shown in the right panel of Figure 4, but the peak positions will still remain unchanged.

Table 1, below, gives the most likely dip values for wave heights up to 4 m, which corresponds to a strong breeze of 6 Bf.

The dip values for a flat sea,  $H_s = 0$  m, are those given in the *Nautical Almanac* [1]. Above, we argued that, if all waves would come with the same height, apparent dip would depend on wave height as  $1'.75 \sqrt{h_0 - H_s/2}$ . It will be useful for practical purposes to keep the formula in this form and find the appropriate factor by which  $H_s$  should be multiplied. As on average 14% of the many wave crests that form the apparent horizon are higher than  $H_s/2$  above median sea level, it may be anticipated that this factor will be greater than  $1/2$ , as argued in the introduction. Indeed, we find that multiplying by 0.72 gives a best overall fit. The rule of thumb is then:

$$\text{apparent dip} = 1'.75 \sqrt{h_0 - 0.72H_s} \quad (14)$$

Of course, this formula applies only as long as the expression under the square root is positive. Under

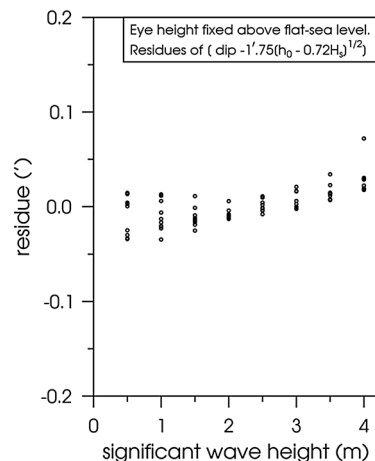


Fig. 5—Residuals of the difference between calculated most probable dip values for fixed eye height and the approximation in Eq. (14).

Table 1—Most probable dip for eye height  $h_0$  and wave height  $H_s$

$h_0 \downarrow; H_s \rightarrow$	0 m	0.5 m	1.0 m	1.5 m	2.0 m	2.5 m	3.0 m	3.5 m	4.0 m
2.5 m	2'.8	2'.5	2'.3	2'.1	1'.8	1'.5	1'.1	0'.8	0'.4
5 m	3'.9	3'.8	2'.6	3'.4	3'.3	3'.1	3'.0	2'.8	2'.6
7.5 m	4'.8	4'.7	4'.5	4'.4	4'.3	4'.2	4'.1	3'.9	3'.8
10 m	5'.5	5'.4	5'.3	5'.2	5'.1	5'.0	4'.9	4'.8	4'.7
15 m	6'.8	6'.7	6'.6	6'.5	6'.4	6'.4	6'.3	6'.2	6'.1
20 m	7'.8	7'.8	7'.7	7'.6	7'.5	7'.5	7'.4	7'.3	7'.3
25 m	8'.8	8'.7	8'.6	8'.6	8'.5	8'.4	8'.4	8'.3	8'.3
Wind $\rightarrow$	0 kn	8 kn	12 kn	14 kn	17 kn	19 kn	20 kn	22 kn	24 kn

the more stringent condition that we limit ourselves to  $h_0 > H_s$ , we find that Eq. (14) reproduces the values of Table 1 well within the usual rounding-off accuracy of 0'.1. This condition is met for all values in the table, with the exception of the largest wave heights at an eye height of 2.5 m. The fit is shown in Figure 5. It includes the data from Table 1 and in addition the corresponding data for eye heights at 12.5, 17.5, and 22.5 m.

### INFLUENCE OF THE CHOICE OF STEEPNESS

Our above calculations have been based on a wave steepness of  $H_s/L = 1/26.2$ , or a wave length 26.2 times the wave height, as found from the article of Groen and Dorrestein [14]. Tucker and Pitt [11] give  $19.7H_s$  for the wave length and from the work of Holthuijsen [12] one finds  $39.2H_s$ . These are the extremes we have found from a non-exhaustive literature search. The influence of this choice must be discussed and we do this by way of two examples.

Table 2 gives apparent dip values for an eye height of 15 m and a significant wave height of 4 m. Compared with a flat sea, the apparent dip has decreased by 0'.67 (Groen and Dorrestein). The difference between the extreme choices of Holthuijsen on the one hand and Tucker-Pitt on the

Table 2—Dip for a fixed eye height of 15 m upon different choices for wave length

$H_s$ (m)	$L/H_s$	comment	apparent dip
0	-	flat sea	6'.78
4	39.2	Holthuijsen [12]	6'.15
4	19.7	Tucker and Pitt [11]	6'.09
4	26.2	Groen and Dorrestein [14]	6'.11
4	[19.7-39.2]	random between 19.7 and 39.2	6'.12

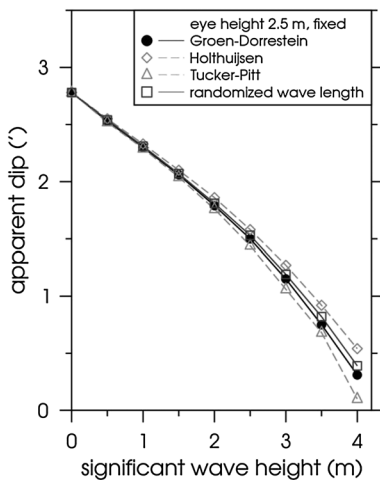


Fig. 6—Apparent dip for 2.5 m eye height, calculated with wave lengths from Groen-Dorrestein [14], Holthuijsen [12], Tucker-Pitt [11] and for wave length randomized between the values of refs. [11] and [12].

other is only 0'.06, between which the Groen-Dorrestein value holds the middle. Randomizing the wave length between the extremes  $19.7H_s$  and  $39.2H_s$  gives an apparent dip that comes to within 0'.01 from the Groen-Dorrestein value.

A second example is shown in Figure 6 for an eye height of 2.5 m, which produces an even more dramatic decrease in dip, but with similar agreement between the four different parameterizations for wave length.

The conclusion seems justified that wave length, according to the Groen-Dorrestein prescription is an appropriate choice for evaluating the effect of wave height on apparent dip. Randomizing wave lengths around this value is probably even more realistic and is shown to yield nearly identical dip values.

### DIP AS MEASURED FROM WAVE CRESTS. THE YACHTSMAN

Above we have discussed the apparent height of the horizon as it is seen from an eye height that is fixed above the median sea level or moving up and down with the waves. As long as the observer does not pay attention to whether he is on a wave crest or in a trough, the effect of the up and down movements of the waters averages out and the most likely value for the apparent horizon dip is the same as for a fixed eye height. A keen yachtsman will, however, attempt to take his sight from the top of a wave, so he can look farther and thus has a better view on what appears to be his horizon.

The wave height distribution of wave crests above median sea level is given by Eq. (9), and the average height is easily found to be  $0.31H_s$ . Thus, if the sailor takes his sight from any arbitrary wave top, he should adopt for his eye height  $h'_0 = h_0 + 0.31H_s$  and use a dip value:

$$\begin{aligned} \text{apparent dip} &= 1'.75 \sqrt{h_0 + 0.31H_s - 0.72H_s} \\ &= 1'.75 \sqrt{h_0 - 0.41H_s} \end{aligned} \quad (15)$$

Not only does this procedure reduce the wave height correction, the correction also becomes better defined. This is illustrated in Figure 7, which shows that the distributions of horizon dip get significantly narrower by selectively taking one's sights from a wave top.

But one can do better: if our yachtsman does not pick just any wave, but instead waits for the highest of the next five that he sees coming, his effective eye height will on average be  $h'_0 = h_0 + 0.52H_s$  and his apparent dip will be

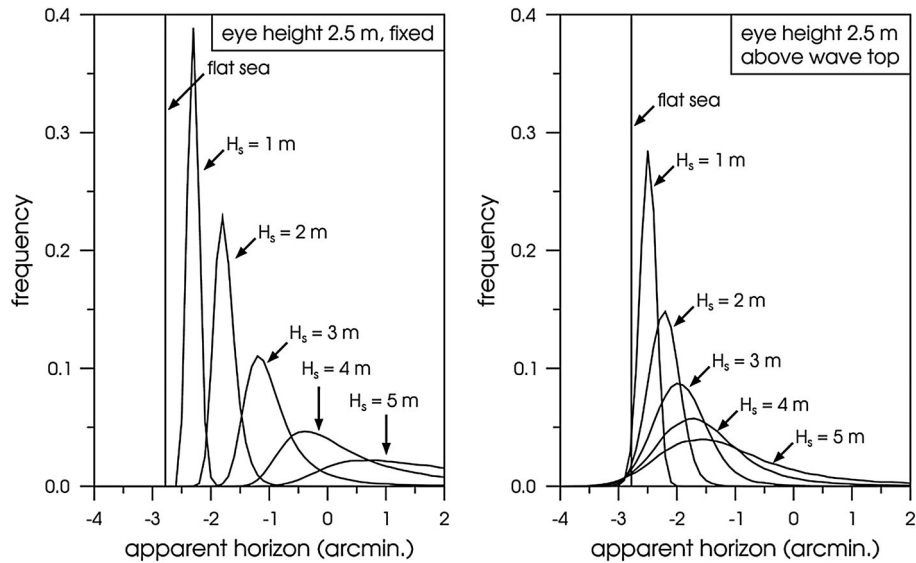


Fig. 7— Showing the improvement of taking sights from a wave top (right hand panel) over a fixed eye height above median sea level (left hand panel).

$$\begin{aligned} \text{apparent dip} &= 1'.75\sqrt{h_0 + 0.52H_s - 0.72H_s} \\ &= 1'.75\sqrt{h_0 - 0.20H_s} \end{aligned} \quad (16)$$

If he wants to do better still, it would take picking the highest out of about 40 waves to reach the break-even point, where the two corrections cancel and he may use his 'uncorrected' eye height.

### CONCLUDING REMARKS AND THE FORMULATION OF A CONJECTURE

When taking the height of a celestial body above the horizon, the navigator must correct his measured height for horizon dip. Nautical handbooks have, throughout history, tabulated dip in an idealized form that supposes the sea to be perfectly flat. Waves have the effect of raising the apparent separation between water and sky. In this work we have evaluated this effect quantitatively for wind-driven waves over deep water and a fully developed sea.

The eye height correction has two distinct contributions. In the first place there is the apparent rise of the horizon itself. It amounts to  $0.72H_s$  and this has to be subtracted from the eye height. The other correction applies only to such cases where sights are selectively taken from wave crests. This is an additive correction to eye height and its value depends on how many wave tops the navigator is willing to let pass before picking the highest one. In the above example, this gave a correction of  $0.52H_s$  for the highest out of five waves. The general result, with a derivation of the relevant formula's is given in the appendix.

The difference with the flat-sea dip values may easily be half or even a full arcminute and for small ships more. Recent tabulations of, for example, the *Nautical Almanac*, give the observables needed for sight reduction to a precision of  $0'.1$ , earlier editions even to one arcsecond, and naval officers were trained to work out their observations to this accuracy. It is therefore remarkable that the much larger effect that waves have on dip seems never to have been considered, at least not in the West.

Today this matter is rather a historical curiosity but the effect may be traced by studying old logbooks. Consider a three-star fix as sketched in Figure 8. The naval officer, or officers, take the

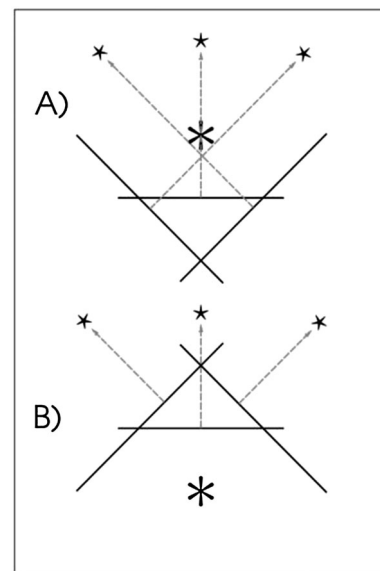


Fig. 8— Three star position fixes, taken from a true position indicated by the asterisk, where in A) the measured heights are systematically too small and too large in B).

heights of three stars and after sight reduction they each draw a position line on the chart. The three position lines enclose a triangle and after approval by the captain, the center of this triangle is accepted as their position. There are two possibilities: as seen from the accepted center, the position lines may be drawn on the same side as the stars, or on opposite sides. We have shown in this work that apparent dip in the presence of waves is smaller than the tabulated dip values. Subtracting a too large dip from the observed height will therefore give too small a height and the position line will be drawn too far backward from the star's foot point on the globe. This is the situation in panel A of Figure 8. Panel B shows the opposite scenario, where the deduced heights are too large.

We conjecture that a historian, whom it amuses to review star fixes from old logbooks, will find that statistically speaking triangles with the triangle pointing downward, as in Figure 8A, occur more often than the ones with the triangle pointing upward.

#### ACKNOWLEDGEMENTS

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#### APPENDIX: Eye Height Correction for Sights from Wave Tops

The distribution of wave heights is given in Eq. (7), which we repeat here:

$$P(H) = \frac{4H}{H_s^2} \exp \left[ -2 \left( \frac{H}{H_s} \right)^2 \right] \quad (\text{A.1})$$

The probability of finding a wave height, not exceeding  $H$ , is

$$\begin{aligned} P(< H) &= \frac{4}{H_s^2} \int_0^H H' \exp \left[ -2 \left( \frac{H'}{H_s} \right)^2 \right] dH' \\ &= 1 - \exp \left[ -2 \left( \frac{H}{H_s} \right)^2 \right] \end{aligned} \quad (\text{A.2})$$

Picking the highest out of a series of  $N$  waves, one may build the distribution function of its height in a straightforward way by combining Eq. (A.1) for the highest wave with Eq. (A.2) for the remaining  $(N-1)$  waves, adding a factor  $N$  for normalization. We shall denote this probability  $P(H > H_2, \dots, H_N)$  as  $P_N(H)$  for short.

$$\begin{aligned} P_N(H) &= N \frac{4H}{H_s^2} \exp \left[ -2 \left( \frac{H}{H_s} \right)^2 \right] \left[ 1 - \exp \left[ -2 \left( \frac{H}{H_s} \right)^2 \right] \right]^{N-1} \\ &= N \frac{4H}{H_s^2} \sum_{k=0}^{N-1} (-)^k \binom{N-1}{k} \exp \left[ -2(k+1) \left( \frac{H}{H_s} \right)^2 \right] \end{aligned} \quad (\text{A.3})$$

This distribution is properly normalized as may be verified by performing the (Poisson-) integrals and evaluating the resulting sum.

$$\int_0^\infty P_N(H) dH = N \sum_{k=0}^{N-1} \binom{N-1}{k} \frac{(-)^k}{(k+1)} = 1 \quad (\text{A.4})$$

The mean of the distribution is found to be:

$$\begin{aligned} \langle H \rangle_N &= \int_0^\infty H P_N(H) dH \\ &= N H_s \sqrt{\frac{\pi}{8}} \sum_{k=0}^{N-1} \binom{N-1}{k} \frac{(-)^k}{(k+1)^{3/2}} \end{aligned} \quad (\text{A.5})$$

As discussed in the text, horizon dip should be read from a tabulation at an effective eye height  $h' = h_0 + \langle H \rangle_N / 2 - 0.72 H_s$ . The first correction,  $\langle H \rangle_N / 2$ , is half the height of the wave from which the sight is taken. The second correction,  $0.72 H_s$ , represents the apparent rise of the horizon for an observer at a fixed height above median sea level and it should be subtracted from one's eye height. Table A.1 below lists the first correction for the largest out of a series of  $N$  waves for different  $N$ .

Table A.1. Additive wave height correction

$N$	$\langle H \rangle_N / 2$ (in units $H_s$ )
1	0.31
2	0.41
3	0.46
4	0.49
5	0.52
10	0.59
20	0.66
50	0.74
100	0.80

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